Toward to usage of regularized Stefan problem solution in icing modeling

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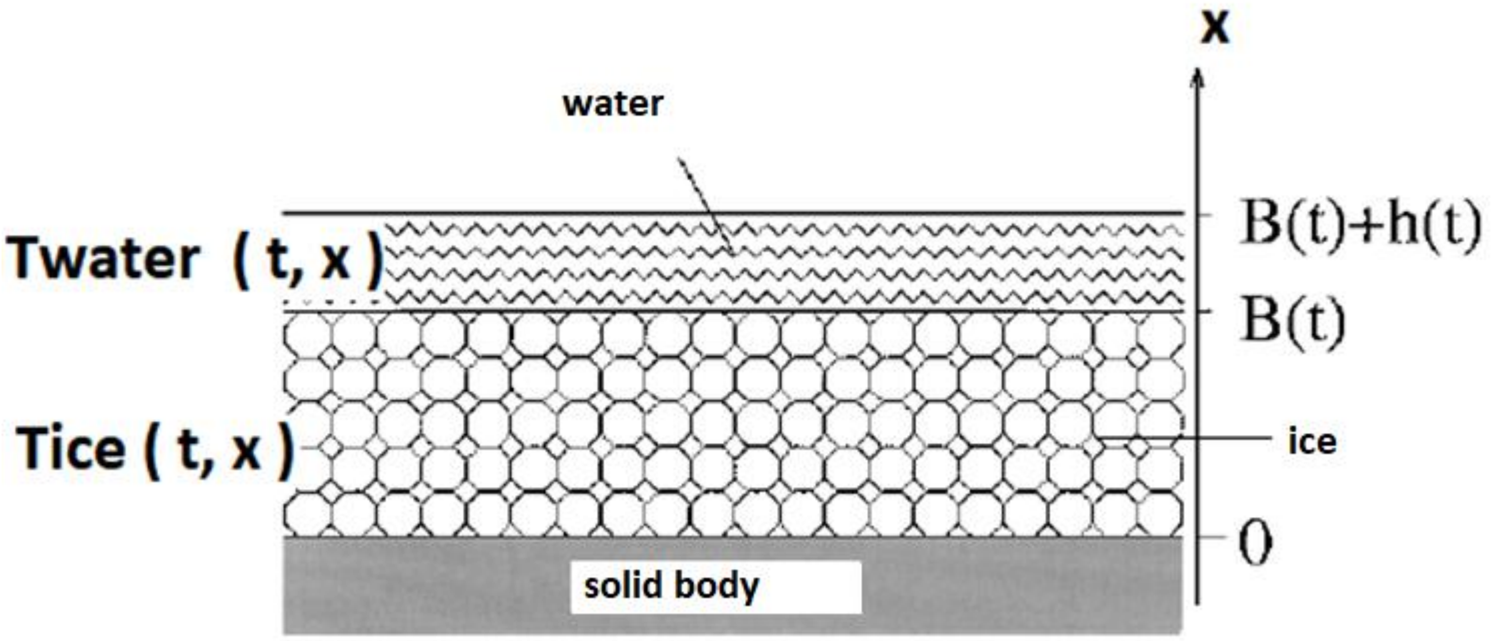
Extended Messenger-Mayers Model for Ice Accretion (1/2)

Scheme of problem



Extended Messenger-Mayers Model. Thermodynamics Relations (2/5)

Governing equations

$$\frac{\partial T_{ice}}{\partial t} = a_{ice}^2 \frac{\partial^2 T_{ice}}{\partial x^2}, \qquad a_{ice}^2 = \frac{k_{ice}}{\rho_{ice} c_{ice}}, \qquad 0 \le x \le B(t); \tag{1}$$

$$\frac{\partial T_{water}}{\partial t} = a_{water}^2 \frac{\partial^2 T_{water}}{\partial x^2}, \quad a_{water}^2 = \frac{k_{water}}{\rho_{water} c_{water}}, B(t) \le x \le B(t) + h(t); \tag{2}$$

$$\rho_{ice} L \frac{\partial B}{\partial t} = k_{ice} \frac{\partial T_{ice}}{\partial x} - k_{water} \frac{\partial T_{water}}{\partial x}, \quad x = B(t) \; ; \tag{3}$$

$$\rho_{ice} \frac{\partial B}{\partial t} + \rho_{water} \frac{\partial h}{\partial t} = \beta W G \; . \tag{4}$$

Here k_{ice}, k_{water} – **coefficients of heat conduction of ice and water,**
ρ_{ice}, ρ_{water} – **densities of ice and water,** c_{ice}, c_{water} – **specific heat coefficiens of ice and water,**
L – **heat of melting,** W -- **velocity of inlet flow,** β – **coefficient of airfoil effectiveness,**
G – **water content in inlet flow**

Extended Messenger-Mayers Model (3/5)

Extended Messenger-Mayers Model

$$\frac{\partial^2 T_{ice}}{\partial x^2} = 0 \,,\quad 0 \le x \le B(t)\,; \qquad \textbf{(1')}$$

$$\frac{\partial^2 T_{water}}{\partial x^2} = 0 \,,\quad B(t) \le x \le B(t) + h(t)\,; \qquad \textbf{(2')}$$

$$\rho_{ice} L \frac{\partial B}{\partial t} = k_{ice} \frac{\partial T_{ice}}{\partial x} - k_{water} \frac{\partial T_{water}}{\partial x} \,,\quad x = B(t) \;; \qquad \textbf{(3)}$$

$$\rho_{ice} \frac{\partial B}{\partial t} + \rho_{water} \frac{\partial h}{\partial t} = \beta W G \,. \qquad \textbf{(4)}$$

Ref: T. G. Myers, Extension to the Messinger Model for Aircraft Icing – AIAA JOURNAL, vol. 39, No. 2, February 2001.

Liu Model (4/5)

Liu Model (2019)

$$\frac{\partial T_{ice}}{\partial t} = a_{ice}^2 \frac{\partial^2 T_{ice}}{\partial x^2}, \quad 0 \le x \le B(t); \tag{1}$$

$$\frac{\partial^2 T_{water}}{\partial x^2} = 0, \quad B(t) \le x \le B(t) + h(t); \tag{2'}$$

$$\rho_{ice} L \frac{\partial B}{\partial t} = k_{ice} \frac{\partial T_{ice}}{\partial x} - k_{water} \frac{\partial T_{water}}{\partial x}, \quad x = B(t) \;; \tag{3}$$

$$\rho_{ice} \frac{\partial B}{\partial t} + \rho_{water} \frac{\partial h}{\partial t} = \beta W G \;. \tag{4}$$

Ref: Tong Liu, Kun Qu, Jinsheng Cai, Shucheng Pan, A three-dimensional aircraft ice accretion model based on the numerical solution of the unsteady Stefan problem, - In: doi.org/10.1016/j.ast.2019.105328

Regularized Stefan Model (5/5)

Regularized Stefan Model

$$\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2} \quad , \quad \begin{array}{l} 0 \le x \le B(t) : \quad a = a_{ice} \ , \\ B(t) < x \le B(t) + h(t) : \quad a = a_{water} \ , \end{array} \qquad \text{(1''-2'')}$$

$h(t)$ => solution of equation $T(t, h(t)) = 0$, **(5)**

$$\rho_{ice} \frac{\partial B}{\partial t} + \rho_{water} \frac{\partial h}{\partial t} = \beta W G \ . \qquad \text{(4)}$$

History Overview of Stefan Problem

- The Stefan problem is a special type of boundary value problem for partial differential equations describing changes the phase state in a substance, in which the position of the phase interface changes with respect to time
- The presence of phase boundaries that are set implicitly and can change with respect to time is a characteristic feature of such problems.
- The velocity of the inter-phase boundary displacement is related with the condition at the interface
- The interface condition was firstly proposed by Joseph Stefan in 1889
- In the modern literature, Stefan problems related with free moving boundary problems and phase transition problems
- Typical examples of the Stefan problem are the problems of ice melting with a changing boundary between the phases of ice and water, as well as the problems of melting a solid substance. These problems considered by various authors in analytical and numerical formulation
- The problem of existence of general solution of Stefan problem studied by prof. O. A. Oleinik and her Ph.D. student S. L.Kamenomosyskaya at the end of fifties and begining of sixties last century

Classic Stefan Problem

Classic Stefan Problem

$$\frac{\partial T_{ice}}{\partial t} = a_{ice}^2 \frac{\partial^2 T_{ice}}{\partial x^2}, \qquad 0 \le x < h(t); \tag{6}$$

$$\frac{\partial T_{water}}{\partial t} = a_{water}^2 \frac{\partial^2 T_{water}}{\partial x^2}, \qquad h(t) \le x < L = +\infty; \tag{7}$$

$$T(0,x) = \begin{cases} C_{ice}, & x = 0 \\ C_{water}, & x > 0 \end{cases}, \tag{8}$$

$$T(t,x) = \begin{cases} C_{ice}, & x = 0, \\ C_{water}, & x = L = +\infty \end{cases}, \tag{9}$$

$$\gamma \frac{dh}{dt} = a_{ice}^2 \frac{\partial T_{ice}}{\partial x} \Big|_{x=h(t)-0} - a_{water}^2 \frac{\partial T_{water}}{\partial x} \Big|_{x=h(t)+0}, \tag{10}$$

$$T_{ice}\big|_{x=h(t)-0} = T_{water}\big|_{x=h(t)+0} \quad . \tag{11}$$

Regularization of Stefan problem

Regularized Stefan problem

$$\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2}, \qquad 0 \le x < L = +\infty;$$

$$a(t,x) = \begin{cases} a_{ice}, & 0 \le x < h(t), \\ a_{water}, & h(t) \le x \le L = +\infty \end{cases},$$

$$T(0,x) = \begin{cases} C_{ice}, & x = 0 \\ C_{water}, & x > 0 \end{cases},$$

$$T(t,x) = \begin{cases} C_{ice}, & x = 0, \\ C_{water}, & x = L = +\infty \end{cases},$$

$h(t) =>$ solution of equation $T(t,h(t)) = 0$.

Here $T=0$ – temperature of phase transition

Numerical Methods Overview

- Analytical and numerical methods are used to solve the Stefan problem
- Analytical methods are possible only for a limited number of cases and simplified problem formulation
- Numerical methods have become more widespread, especially in connection with the development of computer technology.
- In the works of A.A. Samarskiy and B.M. Budak and co-authors [1, 2], for solving Stefan's problems, a procedure for smoothing the coefficients on a certain interval when crossing the phase transition boundary is proposed
- The disadvantage of this approach is the dependence of the solution accuracy on the smoothing parameter and the low accuracy of calculating the position of the phase transition boundary
- In the work of T.A. Bengina, M.Yu. Livshits [3] developed a numerical method for solving a Stefan-type problem based on the transformation of coordinates, transforming a time-varying domain into a stationary one, and a finite-difference solution of the resulting problem
- A.F.Albu and V.I.Zubov [4] proposed an iterative scheme of the first order of approximation in time and the second in space for problems in the enthalpy setting
- P.V. Breslavsky and V.I. Mazhukin in [5] proposed an algorithm for the numerical solution of the Stefan problem with an explicit separation of the phase front on difference grids with dynamic adaptation
- The article by S. L. Borodin [6] provides an overview of the most well-known numerical methods for solving the Stefan problem, as well as the method developed by the author for catching the front into a grid node using an implicit scheme

Hybrid Scheme H2Oder for Regularized Stefan Problem

Euler scheme with 2 iterations is used

$$\frac{T_i^{n+1}-T_i^n}{\tau} = \frac{1}{2}\left(a_i^{n+1/2}\right)^2\{\Delta_i^{n+1} + \Delta_i^n\} \qquad (12)$$

with switching of stensil for evaluation of 2nd oder derivatives only at points of phase transition

$$\begin{gathered} if\ d_i^n > 0\ then \\ if\ \ (T_i^n < 0\ and\ T_{i+1}^n > 0\ )\ then\ \Delta_i^n = \ D_{i-}^n \\ else\ \ \Delta_i^n = \ D_i^n \\ else \\ if\ (\ T_i^n > 0\ and\ T_{i+1}^n < 0\ )\ then\ \Delta_i^n = \ D_{i+}^n \\ else\ \ \Delta_i^n = \ D_i^n \end{gathered}$$

Here

$$d_i^n = \frac{T_{i+1}^n - T_{i-1}^n}{2\Delta x} \qquad (13)$$

$$D_i^n = \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} \qquad (14)$$

$$D_{i+}^n = \frac{2T_i^n - 5T_{i+1}^n + 4T_{i+2}^n - T_{i+3}^n}{(\Delta x)^2} \qquad (15)$$

$$D_{i-}^n = \frac{2T_i^n - 5T_{i-1}^n + 4T_{i-2}^n - T_{i-3}^n}{(\Delta x)^2} \qquad (16)$$

are approximations of the first and second derivatives of temperatute with respect to spatial variable of the second order approximation

Results of Computations (1/2)

- Initial data

  N = 100; dx=0.001;

  Tice = - 20.0;

  Twater = + 5.0;

  tau = 0.36;

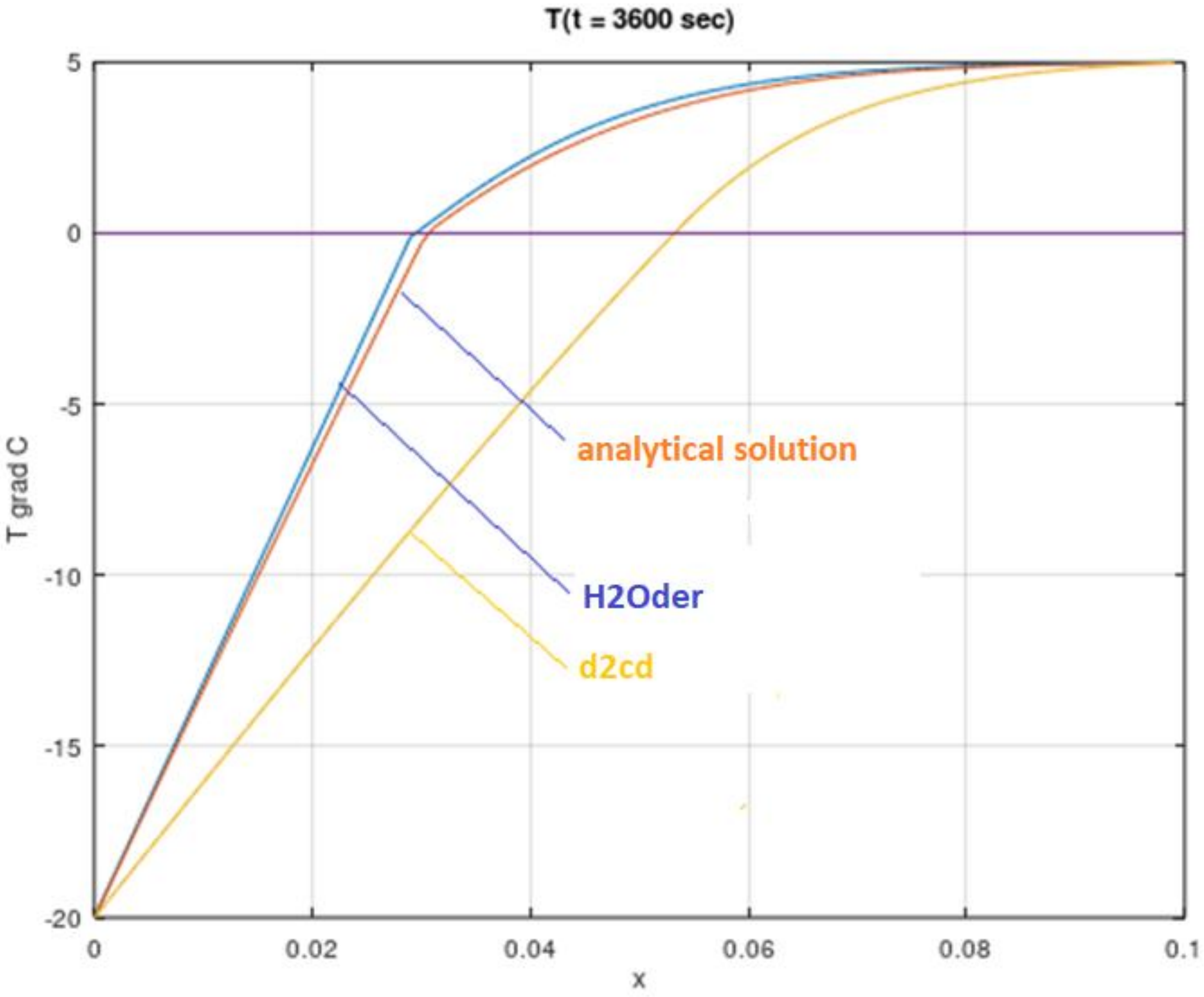
  nTimeSteps=10000;

  aWater=0.000000136;

  aIce =0.00000122

- In d2cd scheme in all grid points

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}$$



Results of Computations (2/2)

- Computation of phase transition boundary is provided by solving implicit non-linear equation

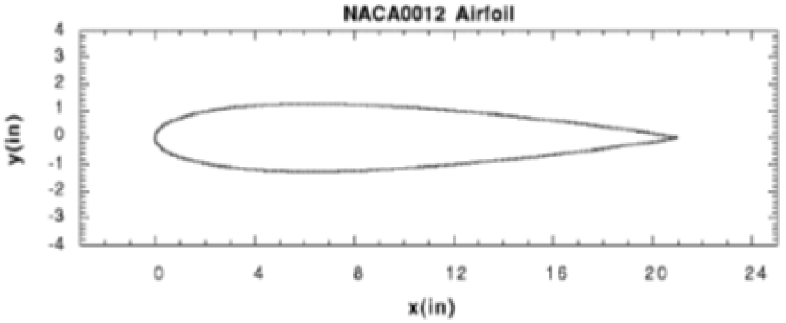
$$T(x_{ice}) = 0$$

at $t = 3600\ s$

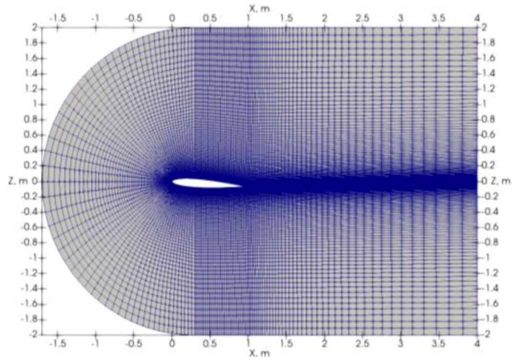
Algorithm	x_{ice}
Analytical solution (A.N.Tikhonov and A.A.Samarskii, Equations of Mathematical Physics, 1972)	0.0307
H2Oder	0.0294
d2cd	0.0532

Results of Computations for NACA0012 using iceFoam and Extended Messenger-Mayers Model

General view of NACA 0012 airfoil

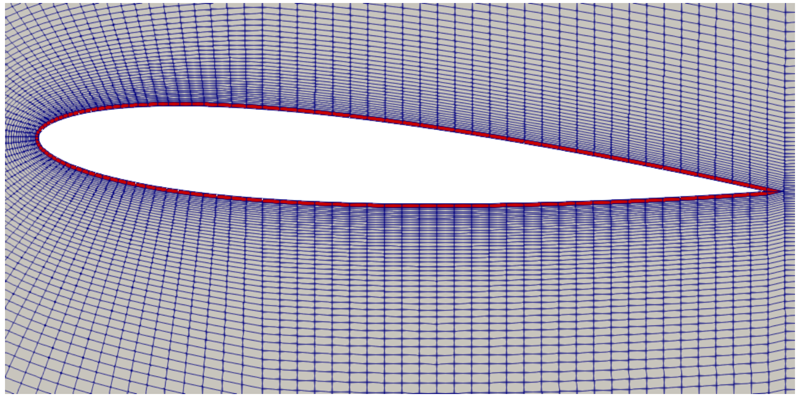


Computational domain and computation grid

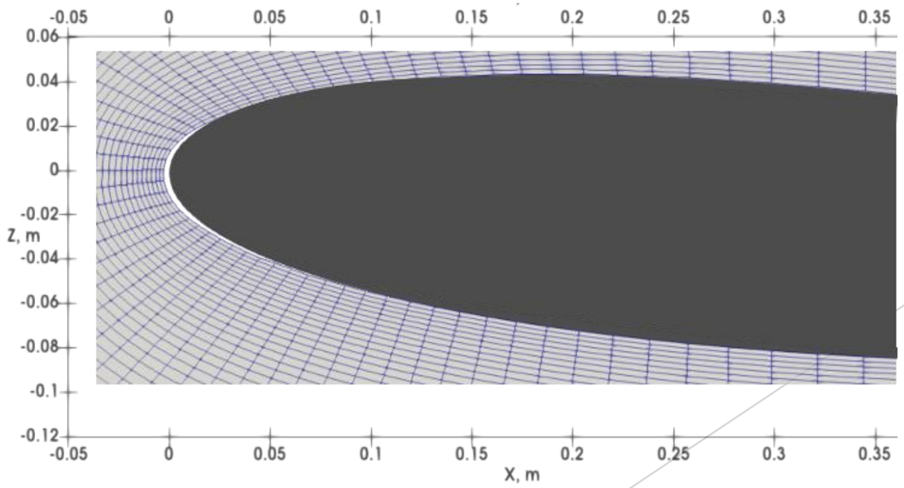


Incoming velocity 102.8 m/s
Temperature 261.52 K
Liquid water content 0.47 g/m^3
Droplets diameter 30 μm
Degree of free-stream turbulence Tu=5%
Number of cells 16000
Number of cells in liquid film 120 * 1
Time step ≈ 1e-5

Computational grid near the airfoil in magnified scale



Ice accretion on the NACA 0012 airfoil computed by iceFoam solver. Time *t*=84 s



Field Observations of Ice Formation

- Boing 737 NEO 800, altitude 11 km, temperature overboard -61 degrees C, 02.28.2020
- Ice forms on the trailing edges of the wings and ends of the wing anti-shock bodies, as well as on the trailing edges of the retracted spoilers (in the gaps between the surfaces of the wings and spoilers) in the so-called bottom areas
- In the bottom areas, sequences of low-intensity reverse vortices appear, similar to the vortices arising in cavern flows and when flowing around a backward-facing step. An analytical solution describing these vortices is given in the book by J. Batchelor "Introduction to fluid dynamics" [8]

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Thanks for Attention!